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A Hypothesis on Atmospheric Heating and its Consequences in Terms of the Sea Breeze

by

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I INTRODUCTION

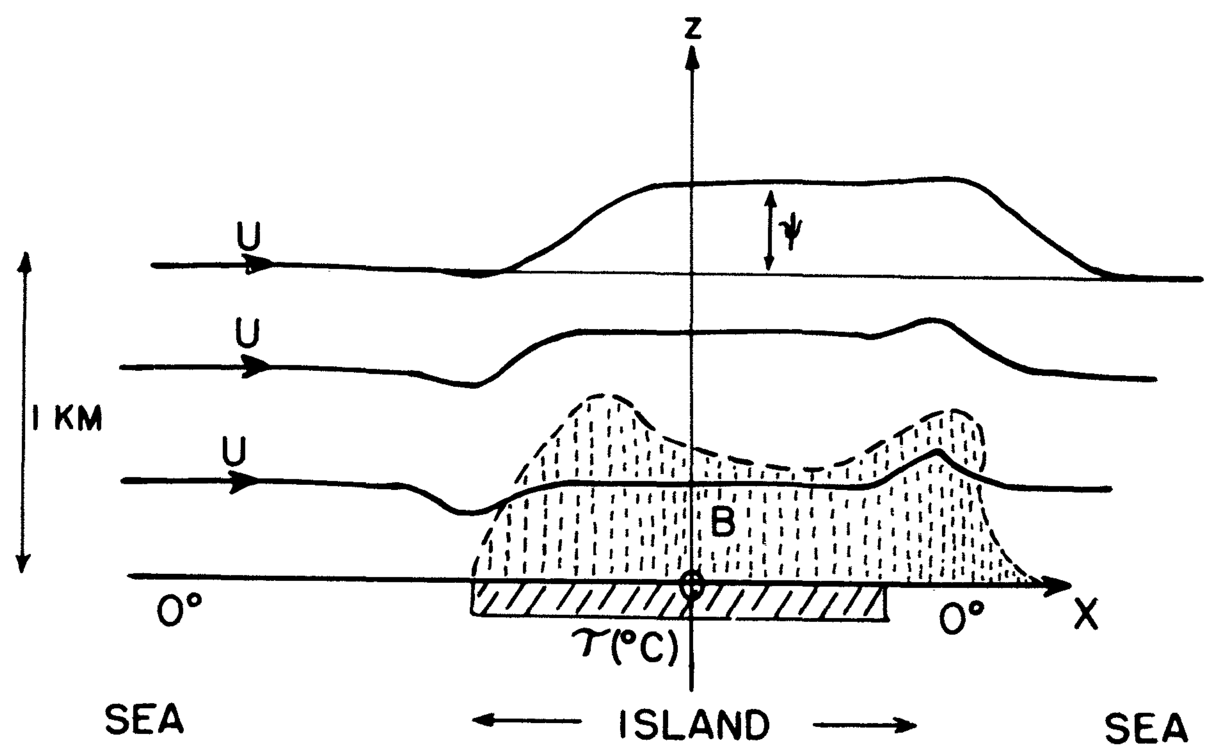
In a series of observational and theoretical studies of the air flow over an island (Malkus and Bunker, 1952), (Malkus and Stern, 1953), (Stern and Malkus, 1953), the connection between the turbulent heating in the lower levels of the atmosphere and the mean velocity perturbations arising from this heating has been brought out. These mean motions may appear as quite sizeable perturbations and at large distances from the energy source, as it is evidenced by the regularly spaced cloud streets that have been observed in the lee of small flat oceanic islands.

However, the problem is of more general interest than explaining local phenomena due to heating. It is hoped that if a satisfactory model of the small scale effects can be obtained, then this may eventually lead to an understanding of larger scale and more important meteorological problems.

A major problem in the previously mentioned theoretical studies was the specification of the gross features of the turbulent heating. It was felt that the conventional eddy conduction equation, which is frequently used to get a first orientation into many problems of atmospheric turbulence, was inadequate in the present type problem. In this paper we shall elaborate on this inadequacy and try to make plausible and extend the formalism used by Stern and Malkus (1953), by recourse to a physical arguement.

It is then sought to test the hypothesis by applying the theory to the well-known phenomenon of the sea breeze. This will be based on the perturbation of an initially undisturbed gradient wind that is perpendicular to a heated coast. The superposition of this gradient wind is not only a realistic condition that has not been investigated theoretically (to the author's knowledge), but insures that the linearization technique is sounder than if it were applied to an atmosphere initially at rest. If the present theory of atmospheric heating is adequate it should give predictions on the sea breeze component which are in agreement with the well known qualitative features, and in addition, it may be expected to lead to specific quantitative conclusions regarding the variation of the sea breeze with the various parameters, which are capable of being checked by observations.

A simple description of the physical processes which produce the mean motions may be obtained by reference to Figure 1, which is a schematic diagram of the air flow over an island that is at a constant temperature above the surrounding water. At large distances upstream the undisturbed flow is taken as adiabatic, with a slight gradient wind (U) perpendicular to the coast. The undisturbed lapse rate and stability are also considered constant. As a result of the large lapse rates that are established over the island, a turbulent ground layer B is developed wherein heat is transferred upwards by eddies. This heating tends to displace the mean streamlines upwards; however, in order to maintain the boundary condition of zero vertical velocity at the ground, there is an opposing displacement due to the stability of the air. The nature and behavior of this component is quite similar to the displacements



<u>Figure 1</u> Streamlines of air flow over an island that is a constant temperature above the surrounding water. The island is infinitely extensive in the y direction (perpendicular to this section).

arising as the result of air flowing over a mountain barrier. Whereas the turbulent heating and the displacements associated with it are confined to the ground layer B (some hundreds of meters in vertical extent), the mountain component is appreciable at points far from B. Hence the mean streamlines outside the ground layer are the same as those that would be produced by an equivalent mountain whose shape depends on the temperature excess of the island, the undisturbed wind and stability, and the vertical extent of B.

II DERIVATION OF THE PERTURBATION EQUATIONS

ξ, η, ζ, t_1 = three rectangular coordinates and time (in c.g.s. units) respectively. (These will be replaced later by the corresponding dimensionless coordinates x, y, z, t.)

U, V = components of undisturbed wind in ξ, η directions.

u', v', w' = components of mean disturbed velocities in ξ, η, ζ directions.

$\bar{p}, \bar{\rho}, \bar{T}$ (or T_m), α, $s = \frac{\Gamma - \alpha}{T_m}$ = undisturbed pressure, density, absolute temperature, lapse rate, and stability.

p', ρ', T' = perturbation pressure, density, and temperature.

$D_1 = \frac{\partial}{\partial t_1} + U\frac{\partial}{\partial \xi} + V\frac{\partial}{\partial \eta}$ linearized total derivative.

$f = 2\omega \sin \Omega$ = Coriolis parameter.

The axis will be oriented so that ξ is in the direction of the horizontal temperature gradient at the ground, and the ζ axis points vertically upwards. On performing a first order perturbation of the inviscid hydrodynamic equations and the equation of state, one obtains the following basic equations:

(1) $$D_1 u' - fv' = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial \xi}$$

(2) $$D_1 v' + fu' = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial \eta}$$

(3) $$D_1 w' = -\frac{1}{\bar{\rho}} \frac{\partial p'}{\partial \zeta} - \frac{\rho'}{\bar{\rho}} g$$

(4) $$\frac{\partial u'}{\partial \xi} + \frac{\partial v'}{\partial \eta} + \frac{\partial w'}{\partial \zeta} = 0$$

(5) $$p' = \bar{\rho} R T' + \rho' R \bar{T}$$

Equation (3) is now used to eliminate the pressure from (1) and (2) and equation (4) is written in terms of the displacement functions ψ and φ. (Subscripts denote partial derivatives.)

(6) $$D_1 u'_\zeta - fv'_\zeta = \frac{g}{\bar{\rho}} \rho'_\xi + D_1 w'_\xi$$

(7) $$D_1 v'_\zeta + fu'_\zeta = D_1 w'_\eta$$

(8) $$u' = -U(\psi_\zeta + \varphi_\eta); \quad w' = U\psi_\xi ; \quad v' = U\varphi_\xi$$

Substitute (8) in (6) and (7).

(9) $$-D_1 (\psi_{\zeta\zeta} + \varphi_{\eta\zeta}) - f\varphi_{\xi\zeta} = \frac{g}{U\bar{\rho}} \rho'_\xi + D_1 \psi_{\xi\xi}$$

(10) $$D_1 \varphi_{\xi\zeta} - f(\psi_{\zeta\zeta} + \varphi_{\eta\zeta}) = D_1 \psi_{\xi\eta}$$

From (5) it follows that

$$\frac{1}{\bar{\rho}} \rho' = -\frac{T'_\xi}{\bar{T}} + \frac{p'_\xi}{\bar{\rho} R \bar{T}}$$

It may be shown (see Malkus and Stern, 1953) that the second term on the right hand side leads to a small damping term in the final differential equation, which only effects the amplitude of the displacement at large altitudes. The same is true for the terms involving $\frac{\partial \bar{\rho}}{\partial \zeta}$, which have been omitted in the previous elimination.

To investigate the influence of these damping terms the reader is referred to the literature on the mountain wave problem (e.g., Scorer, 1949). These terms are eliminated at the outset of the present discussion by writing

$$(11) \qquad \frac{1}{\bar{\rho}} \rho'_{\xi} = - \frac{1}{\bar{T}} T'_{\xi}$$

If the flow were adiabatic the temperature perturbation could be eliminated from the above equations by utilizing the fact that the total derivative of the potential temperature (θ) was zero. In the present problem, however, an amount of thermal energy is being added at each point which is equal to the divergence of the eddy flux of heat. If this quantity is denoted by $c_p \, H(\xi, \eta, \zeta, t_1)$, where c_p is the specific heat at constant pressure, then the first law of thermodynamics may be written as

$$H = \frac{T}{\theta} \frac{d\theta}{dt_1}$$

The first order perturbation leads to

$$(12) \qquad H = D_1 T' + w'(\Gamma - \alpha) = D_1 T' + U(\Gamma - \alpha) \psi_{\xi} \quad .$$

Equations (11) and (12) are now used to eliminate ρ' and T' from (9).

$$(13)\qquad D_1^2\nabla_1^2\psi + gs\psi_{\xi\xi} + D_1^2\varphi_{\eta\zeta} + fD_1\varphi_{\xi\zeta} = \frac{g}{UT_m}\frac{\partial H}{\partial \xi}$$

$$(14)\qquad -f\psi_{\zeta\zeta} - D_1\psi_{\xi\eta} + \varphi_{\eta\zeta} + D_1\varphi_{\xi\zeta} = 0$$

where T, the mean undisturbed absolute temperature has been written as T_m to avoid confusion with subsequent notation, and $\nabla_1^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \zeta^2}$.

At this point the dimensionless coordinates (x, y, z, t) will be introduced, replacing (ξ, η, ζ, t_1).

$$(15)\qquad x = \xi\sqrt{\frac{gs}{U^2}} \quad y = \eta\sqrt{\frac{gs}{U^2}} \quad z = \zeta\sqrt{\frac{gs}{U^2}} \quad t = t_1\sqrt{gs}$$

In addition: $c^2 = \frac{f^2}{gs}$; $D = (\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y})$;

$$\nabla^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2});\qquad \gamma = \frac{V}{U}$$

Then (13) and (14) become:

$$(16)\qquad D^2\nabla^2\psi + \psi_{xx} + D^2\varphi_{yz} + cD\varphi_{xz} = \frac{1}{sT_m\sqrt{gs}}\frac{\partial H}{\partial x}$$

$$(17)\qquad -c\psi_{zz} - D\psi_{xy} - \varphi_{yz} + D\varphi_{xz} = 0$$

With a constant value of the Coriolis parameter the elimination of φ from (16) and (17) is quite simple. Since the x-axis was chosen in the direction of the horizontal temperature gradient ψ_y may be omitted from the final differential equation.

$$(18)\qquad \boxed{(\frac{\partial}{\partial t} + \frac{\partial}{\partial x})^2\nabla^2\psi + \psi_{xx} + c^2\psi_{zz} = \frac{1}{sT_m\sqrt{gs}}\frac{\partial H}{\partial x}}$$

When ψ and H are written in terms of Fourier expansions whose typical terms are $e^{ikx}\, e^{i\lambda t}\,\bar{\psi}(z)$ and $e^{ikx}\, e^{i\lambda t}\,\bar{H}(z)$ respectively (henceforth bars above letters indicate Fourier components), then equation (18) becomes

$$(19) \qquad \left[c^2 - (k+\lambda)^2\right] \frac{\partial^2 \bar{\psi}}{\partial z^2} - \left[k^2 - k^2\,(k+\lambda)^2\right] \bar{\psi} = \frac{ik\bar{H}}{sT_m \sqrt{gs}}$$

The elimination of φ from (16) and (17) when the latitudinal change of the Coriolis parameter is considered, is more difficult. This has been carried out for the case when the horizontal temperature gradient along the ground is in the east-west direction, so that $\frac{\partial f}{\partial \xi} = 0$ and $\frac{\partial f}{\partial \eta} = \beta$ (or in dimensionless units $\frac{\partial c}{\partial y} = \beta_o$ where $\beta_o = \frac{\beta U}{gs}$). The result corresponding to (19) is

$$(20) \qquad \left[\frac{c^2 - (k+\lambda)^2}{1 - \frac{\beta_o}{k(k+\lambda)}}\right] \frac{\partial^2 \bar{\psi}}{\partial z^2} - \left[k^2 - k^2\,(k+\lambda)^2\right] \bar{\psi} = \frac{ik\bar{H}}{sT_m \sqrt{gs}}$$

III THE DESCRIPTION OF THE HEATING FUNCTION

If H and λ are set equal to zero equations (19) and (20) give solutions for the steady state mountain problem (Queney, 1947, 1948). However, the inhomogeneous term (H) is the essential driving mechanism which produces the mean perturbations over the flat island, and it must be specified by another relation before these equations can be solved.

In order to describe the turbulent heating one might be tempted to use the conventional equation of eddy conduction and set the total derivative of the potential temperature equal to a constant times the second derivative of the temperature with respect to height. It is well known that although the eddy conductivity varies considerably in space and time, it is often possible to obtain a first orientation into many phenomena by assuming an effective constant value for this quantity. It is contended however, that in the present type of problem where the heating induces non-negligible mean vertical velocities, this approach is inapplicable, at least without considerable modification. To give a simple illustration of this inadequacy, consider the essentially adiabatic waves which occur outside the ground layer, e.g., lee waves. These waves vary in magnitude in the vertical as well as the horizontal direction and the second derivative of temperature with height is not zero. In fact, application of the simple eddy conduction equation would lead to the untenable conclusion that the amount of heat added at a point far in the lee may be comparable with the heat that is supplied at a point in the middle of the turbulent ground layer. If one is to use the ideas of eddy conduction at all, it is necessary to distinguish between the temperature gradients that are maintained by turbulent transport and those that are due to adiabatic convective motions. This difficulty might, in principle, be surmounted by using the conventional eddy equation with a conductivity that was an order of magnitude less outside B than inside, but because of the complicated shape of B, this formalism seems hopelessly complex to introduce into the hydrodynamic equations. On the other hand, an adequate formalism must insure that the heating function approaches zero at far distances from the island, even when mean vertical velocities due to the heating exist in these regions (it is assumed, of course, that there are no other mechanisms, such as condensation, which are producing heat sources). The following discussion is an extension and elaboration of the method of Stern and Malkus (1953).

Because of the fact that the temperature perturbation is due to a combination of two different mechanisms; namely, turbulent transport and adiabatic convective motions, it is inconvenient to attempt to describe the heating in terms of this quantity as the dependent variable. Instead, the present formalism evolves about the heating function which is the divergence of the eddy flux, and we shall try to justify the major premise that this can be determined independently of the mean vertical velocities which it produces. This is clearly the case at the lower boundary, as is seen from Eq. 12, and here the heating function can be determined from the temperature along and the shape of this boundary, and is not explicitly related to the vertical velocities that are produced aloft. What can be said about the variation of the heating function in the vertical?

In Figure 1, consider the effect of inserting a series of horizontal grids, or large screens, into the field of motion. At any point the spacing of the gridwork is large compared with the mean eddy size and small compared to the distance over which one averages to obtain the mean vertical velocity and temperature. Then the turbulent eddies will pass through the gridwork relatively unaffected while the mean vertical velocities will be reduced towards zero. The screens are effectively a solid barrier to the mean motions. It is hypothesized that the eddy flux of heat, and in particular, its divergence, is unaffected by variations of the mean perturbations due to the imposition of these constraints. By this means it is possible to consider the extremely complex turbulent heating process independently; then, acting as a fixed driving force, the heating function produces streamline displacements as determined by Equation 18, and the boundary conditions.

Accepting this hypothesis, a differential equation for H is now derived by applying the simple ideas of eddy conduction to the model in which the mean vertical velocities have been reduced to zero. Denote the temperature in this model by T_e (ξ, η, ζ, t) and note that this is, in general, different from T' (ξ, η, ζ, t), the temperature distribution in the model whose mean motions it is desired to investigate (Figure 1). To see this, consider the region outside B where $H \to 0$. For the original model (Figure 1, flat island) it follows from (12) that

$$w'(\Gamma - \alpha) + U\frac{\partial T'}{\partial \xi} + V\frac{\partial T'}{\partial \eta} + \frac{\partial T'}{\partial t_1} = 0 ,$$

whereas

$$w_e' = 0 \quad \text{and} \quad U\frac{\partial T_e}{\partial \xi} + V\frac{\partial T_e}{\partial \eta} + \frac{\partial T_e}{\partial t_1} = 0$$

for the second model.

Applying the first law of thermodynamics to the second model, we obtain the following relations:

(21) $$H = \frac{\partial T_e}{\partial t_1} + U\frac{\partial T_e}{\partial \xi} + V\frac{\partial T_e}{\partial \eta}$$

(22) $$H = K\frac{\partial^2 T_e}{\partial \zeta^2}$$

Upon elimination of T_e there results

(23) $$\frac{\partial H}{\partial t_1} + U\frac{\partial H}{\partial \xi} + V\frac{\partial H}{\partial \eta} = K\frac{\partial^2 H}{\partial \zeta^2}$$

In deriving (23) it is assumed that K has a constant effective value for the turbulent ground layer B. The fact that the measured eddy conductivity is an order of magnitude less, far outside B is of little moment in this formalism because (23) insures that H is small at these points. It is to be noted that when the mean vertical velocities can be neglected (23) reduces to the familiar eddy conduction equation, where temperature replaces H.

In addition, the boundary value of H follows directly from the first law and is independent of any previous assumptions regarding the eddy transport[1].

(24) $$H(\xi, \eta, o, t_1) = \left(\frac{\partial}{\partial t_1} + U\frac{\partial}{\partial \xi} + V\frac{\partial}{\partial \eta}\right) T'(\xi, \eta, o, t_1)$$

Returning to the model in which the horizontal temperature gradient at the ground is in the direction of the x-axis and writing (23) and (24) in terms of the dimensionless coordinates there results,

---

1 If the ground is not flat but has an elevation given by $\psi(\xi, \eta, o)$, then one merely adds the term $U(\Gamma - \alpha)\frac{\partial \psi}{\partial \xi}$ to the right hand side of (24).

$$(25)\qquad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) H = b^2 \frac{\partial^2 H}{\partial z^2}$$

$$H(x,o,t) = \sqrt{gs}\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) T'(x,o,t)$$

where

$$b^2 = \frac{K}{U^2}\sqrt{gs}$$

Substituting the Fourier components for H and T' we get,

$$\frac{\partial^2 \overline{H}}{\partial z^2} - \frac{i(k+\lambda)}{b^2}\overline{H} = 0$$

$$\overline{H}(z=o) = \sqrt{gs}\ i(k+\lambda)\ \overline{T}_o$$

$$(26)\qquad \therefore \overline{H} = \sqrt{gs}\ i(k+\lambda)\ \overline{T}_o \exp - \left[(\pm i)^{1/2} |k+\lambda|^{1/2} b^{-1} z\right]$$

where

$\overline{T}_o$ is the amplitude of the (k,λ) harmonic of the temperature at the ground and $(\pm i) = e^{\frac{\pm \pi i}{4}}$, according as $(k+\lambda)$ is greater or less than zero.

IV THE GENERAL SOLUTION OF THE DIFFERENTIAL EQUATION FOR THE MEAN STREAMLINES

Upon substituting (26) into (20) there results,

$$(27)\qquad \left[\frac{c^2 - (k+\lambda)^2}{1 - \frac{\beta}{k(k+\lambda)}}\right] \frac{\partial^2 \overline{\psi}}{\partial z^2} - \left[k^2 - k^2 (k+\lambda)^2\right] \overline{\psi} = \frac{-k(k+\lambda)}{s\overline{T}_m} \overline{T}_o \cdot$$

$$\cdot \exp - \left[(\pm i)^{1/2} |k+\lambda|^{1/2} b^{-1} z\right]$$

Solving, one obtains

$$\bar{\psi} = \bar{M}(k,\lambda) \left[h(z,k,\lambda) - f(z,k,\lambda) \right] \tag{28}$$

where

$$\bar{M}(k,\lambda) = \frac{\bar{T}_o}{sT_m} \; \frac{k(k+\lambda)\left[1 - \frac{\beta_o}{k(k+\lambda)}\right]}{\left[(k+\lambda)^2 - c^2\right]\left[\frac{i(k+\lambda)}{b^2} - r^2\right]} \tag{29}$$

and

$$r^2 = \frac{k^2 - k^2\,(k+\lambda)^2}{c^2 - (k+\lambda)^2} \left(1 - \frac{\beta_o}{k(k+\lambda)}\right)$$

$$f(z,k,\lambda) = \text{linear combination of } e^{-rz} \text{ and } e^{+rz}$$

$$h(z,k,\lambda) = \exp - \left[(\pm i) \; |k+\lambda|^{1/2} b^{-1} z \right]$$

The solution for the displacement ψ for an arbitrary temperature distribution along the ground is

$$\psi(x,z,t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \bar{M}(k,\lambda)\, e^{ikx} \; e^{i\lambda t} \left[h(z,k,\lambda) - f(z,k,\lambda)\right] dk\,d\lambda \tag{30}$$

The solution is separated into two parts ψ_1 and ψ_2, where $\psi = \psi_1 - \psi_2$ and

$$\psi_1(x,z,t) = \int_{-\infty}^{+\infty} d\lambda\, e^{i\lambda t} \int_{-\infty}^{+\infty} \bar{M}(k,\lambda)\, e^{ikx}\, h(z,k,\lambda)\, dk \tag{31}$$

$$\psi_2(x,z,t) = \int_{-\infty}^{+\infty} d\lambda\, e^{i\lambda t} \int_{-\infty}^{+\infty} \bar{M}(k,\lambda)\, e^{ikx}\, f(z,k,\lambda)\, dk \tag{32}$$

The first component satisfies the heat conduction equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\psi_1 = b^2 \frac{\partial^2 \psi_1}{\partial z^2}$$

while the second component satisfies the homogeneous differential equation. Since $\psi(x,o,t) = 0$ and $h(o,k,\lambda) = 1$. Therefore,

$$(33)\qquad \boxed{\psi_2\,(x,o,t) = M(x,t) = \int_{-\infty}^{+\infty} e^{i\lambda t}\,d\lambda \int_{-\infty}^{+\infty} \bar{M}(k,\lambda)\, e^{ikx}\,dk}$$

M(x,t) has been called the equivalent mountain function since the component $\psi_2(x,z,t)$ is mathematically identical to the air flow over a mountain whose profile is given by (33). In addition, (33) also determines the conduction component ψ_1, but aside from the fact that it is necessary to satisfy the boundary condition at the ground it is of little interest, since it decreases rapidly outside the ground layer. Thus the problem of air flow over heated terrain is essentially reduced to the investigation of the equivalent mountain (Equation 33). In the discussion to follow it is convenient to consider the scale of the heating in a fashion similar to that used by Queney (1948) in discussing mountain waves. For each scale different sets of parameters become important and allow simplification of the results. The scale divisions and the approximations entailed in each are summarized on the following page.

Scale	Size of "island" L (k m)	Simplifications in evaluating equivalent mountain
Small	$L \ll \frac{2\pi}{c} \cdot \frac{1}{\sqrt{\frac{gs}{U^2}}}$ $L \ll \frac{2\pi}{\lambda} \cdot \frac{1}{\sqrt{\frac{gs}{U^2}}}$	$\frac{\lambda}{k} \ll 1$ $\frac{c}{k} \ll 1$ Coriolis parameters (c, β_o) omitted from equivalent mountain
Middle	$L \sim \frac{2\pi}{c} \cdot \frac{1}{\sqrt{\frac{gs}{U^2}}}$ and $\frac{2\pi}{\lambda} \cdot \frac{1}{\sqrt{\frac{gs}{U^2}}}$	$k^2 \ll 1$ β_o omitted
Large	$\frac{2\pi}{\lambda} \frac{1}{\sqrt{\frac{gs}{U^2}}} \ll L \lesssim \frac{2\pi}{\beta_o} \frac{\lambda}{\sqrt{\frac{gs}{U^2}}}$	$\frac{k}{\lambda} \ll 1$ $\frac{k}{c} \ll 1$

V THE SMALL SCALE SOLUTION - PREDICTIONS ON THE SEA BREEZE

In this paper, only the small scale solution will be considered. The omission of a discussion of the larger scale solution is not primarily due to any inherent mathematical problems, but rather to the conceptual difficulties in assigning values for the eddy conductivity and the undisturbed parameters. Moreover, it is felt that confidence in the heating hypothesis should first be obtained by applying the theory to such well known local phenomena as the sea breeze to test its adequacy. However, formal manipulation of the larger scale solutions have been carried through and it is believed that these may be of heuristic value. The author intends to be able to report on this in the near future.

For the sake of definiteness we consider a small flat island (of the order of tens of kilometers in width and sensibly infinite in length) which is at a uniform temperature above the surrounding water. The temporal variation of this temperature difference is assumed to be represented by a finite number of Fourier harmonics, with the predominant one being the diurnal period (P = 24 hrs), then $\lambda = \frac{2\pi}{P\sqrt{gs}}$ $(\lambda \sim 10^{-2})$. We now make the small scale approximation and neglect $\frac{\lambda}{k}$ and $\frac{c}{k}$ in comparison with unity. This is valid for heat sources whose characteristic horizontal dimension L is much less than UP. Equation (29) then becomes

$$\overline{M} = \frac{\overline{T}_o}{sT_m} \frac{1}{\frac{ik}{b^2} + 1 - k^2}$$

If the ground temperature is given by

$$T_o(x,t) = \cos \lambda t \int_{-\infty}^{+\infty} \overline{T}_o(k)\, e^{ikx}\, dk$$

then the equivalent mountain is, from (33),

(34) $$M(x,t) = \frac{\cos\lambda t}{sT_m} \int_{-\infty}^{+\infty} \frac{e^{ikx}\, T_0(k)}{\frac{ik}{b^2} - k^2 + 1}\, dk$$

This differs from the steady state solution obtained by Stern and Malkus (1953) only in the presence of the $\cos\lambda t$ term. By simple extension of their results it is easily shown that the sea breeze component perpendicular to the windward shore of the island is

(35) $$u' = \frac{\Upsilon \cos\lambda t}{sT_m} \; \frac{\sqrt{gs}\, \cos z}{\pi(a_2 - a_1)} \int_{-a_1 x}^{-a_2 x} \frac{e^{-p}}{p}\, dp \; ; \quad x \leq 0$$

where

$x = 0$ is at the windward shore

$$\left.\begin{matrix} a_1 \\ a_2 \end{matrix}\right\} = \frac{1}{2b^2}\left[1 \pm \sqrt{1 - 4b^4}\right]$$

$$b^2 = \frac{K}{U^2}\sqrt{gs}$$

$\Upsilon \cos\lambda t$ = temperature excess of the island

In Haurwitz's (1947) heuristic model of the sea breeze it turns out that without friction the predicted sea breeze is 90° out of phase with the temperature. Defant's (1951) model of the sea breeze shows a phase difference of 4.7 hours when friction is not considered. His theory while somewhat similar to the present one uses the conventional law of eddy conduction and further, neglects completely the non-linear inertial term. It is seen from the preceding analysis that the predicted sea breeze component is in phase with the diurnal temperature wave, and it is not necessary to introduce friction to explain this often observed phenomenon.[2]

Figure 2 (after Stern and Malkus, 1953) shows the variation of the sea breeze component (u') at the windward shore with the other atmospheric parameters. The quantities u' and τ in this diagram may be interpreted as instantaneous values in the 24 hour cycle. From this diagram it is seen that the ratio of the eddy conductivity to the square of the basic current is of great importance in determining the magnitude of the sea breeze. It will be noted that, on the scale considered, the Coriolis force has no influence on u' and this is due to the fact that the horizontal pressure gradient and the inertial forces are much larger and control the motion in the x-z plane. However, the sea breeze component parallel to the coast is primarily balanced by the Coriolis force, as shall be seen subsequently when the hodograph is discussed. Before turning to this, some observational work in connection with the preceding formula will be discussed.

The success of the present model in obtaining a realistic picture of the sea breeze phenomenon has encouraged attempts to obtain quantitative observational checks. By differentiating equation (35) with respect to x and denoting

$\sqrt{\frac{gs}{U^2}}\ \frac{\partial u'}{\partial x} = \frac{\partial u'}{\partial \xi}$ at x = 0, z = 0 by G, one arrives at the following

[2]However it must be pointed out that this has only been shown for a model in which the ground temperature reaches its maximum value in a distance that is less than UP.

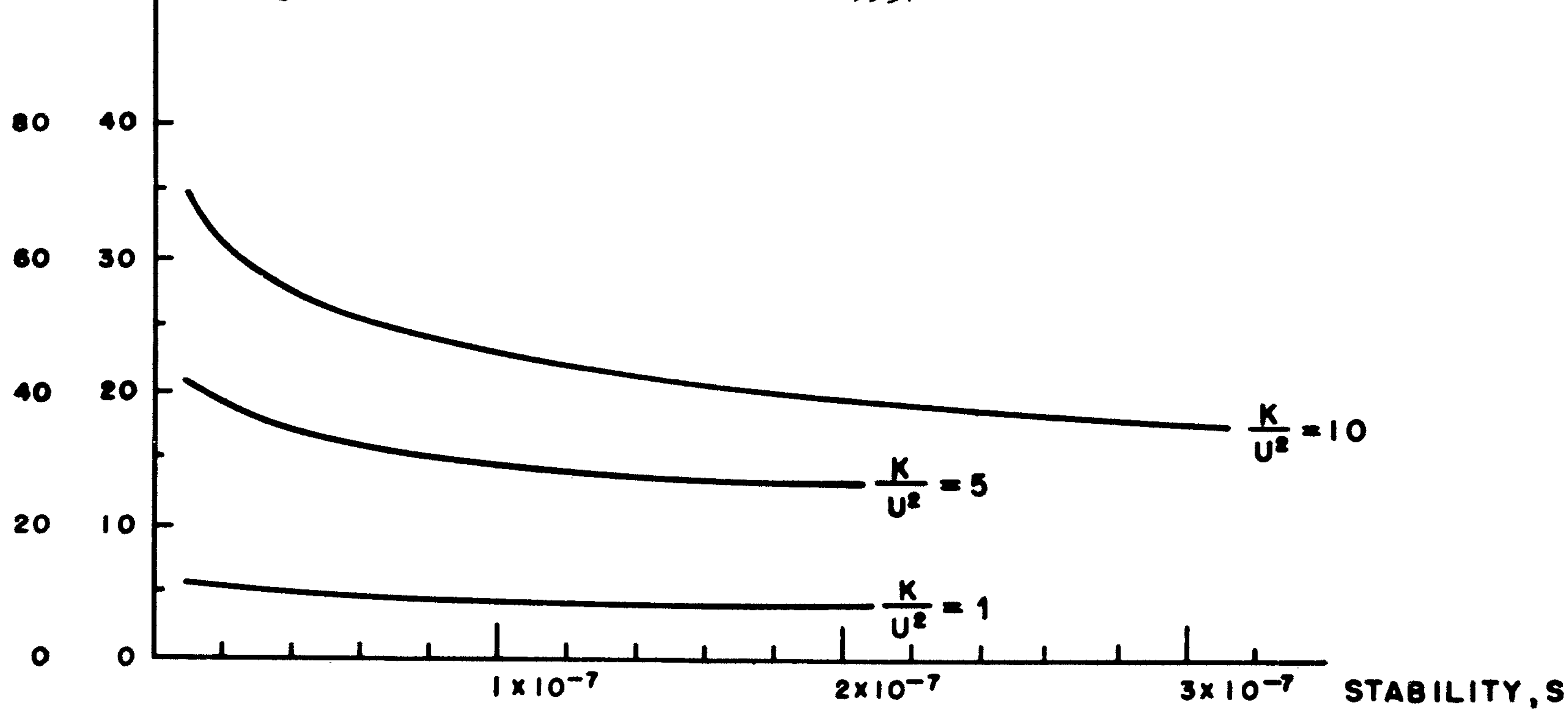


Figure 2 Variation of the amplitude of the sea breeze at the windward shore as a function of the undisturbed stability, S (cm-1).

The ordinate is in units of $\frac{u' \pi T_m}{2 g \tau}$ (sec) while the adjacent scale shows the value u' must have in cm/sec for a particular temperature excess (τ = 0.95°C). K is the effective eddy conductivity of the ground layer (cm^2 sec^{-1}) and U (cm/sec) is the undisturbed wind speed. (After Stern and Malkus, 1953).

simple expression:

$$(36) \qquad \frac{\tau g}{\pi G T_m U} = 1$$

where

τ = temperature excess of island at a given time,

U = undisturbed wind component perpendicular to coast,

T_m = mean absolute atmospheric temperature.

Although all explicit reference to the vertical distribution of heating has been eliminated in the above equation, its validity depends on the fact that a constant effective eddy conductivity for B may be chosen. In addition it contains relatively simple observables (temperature and horizontal wind speed) and seems capable of being checked. Its validity would not only confirm the theory, but would afford a simple means of calculating K (see Stern and Malkus, 1953), by means of an additional formula.

During the summer of 1952, a small exploratory field program was undertaken, mainly with the purpose of determining whether or not the expected slope in the wind profile indicated by (36) could be measured. Figure 3 shows a series of wind profiles obtained running from the southern shore of Nantucket to the interior of the island. Three minute averages on a small cup anemometer placed about seven feet above the ground were made at each station. Although measurements of distance and angles were rough, determination of the mean slope for each of the runs was made and the quantity $\frac{\tau g}{\pi G T_m U}$ computed. The results are shown in Table 1. It is to be pointed out that for comparison

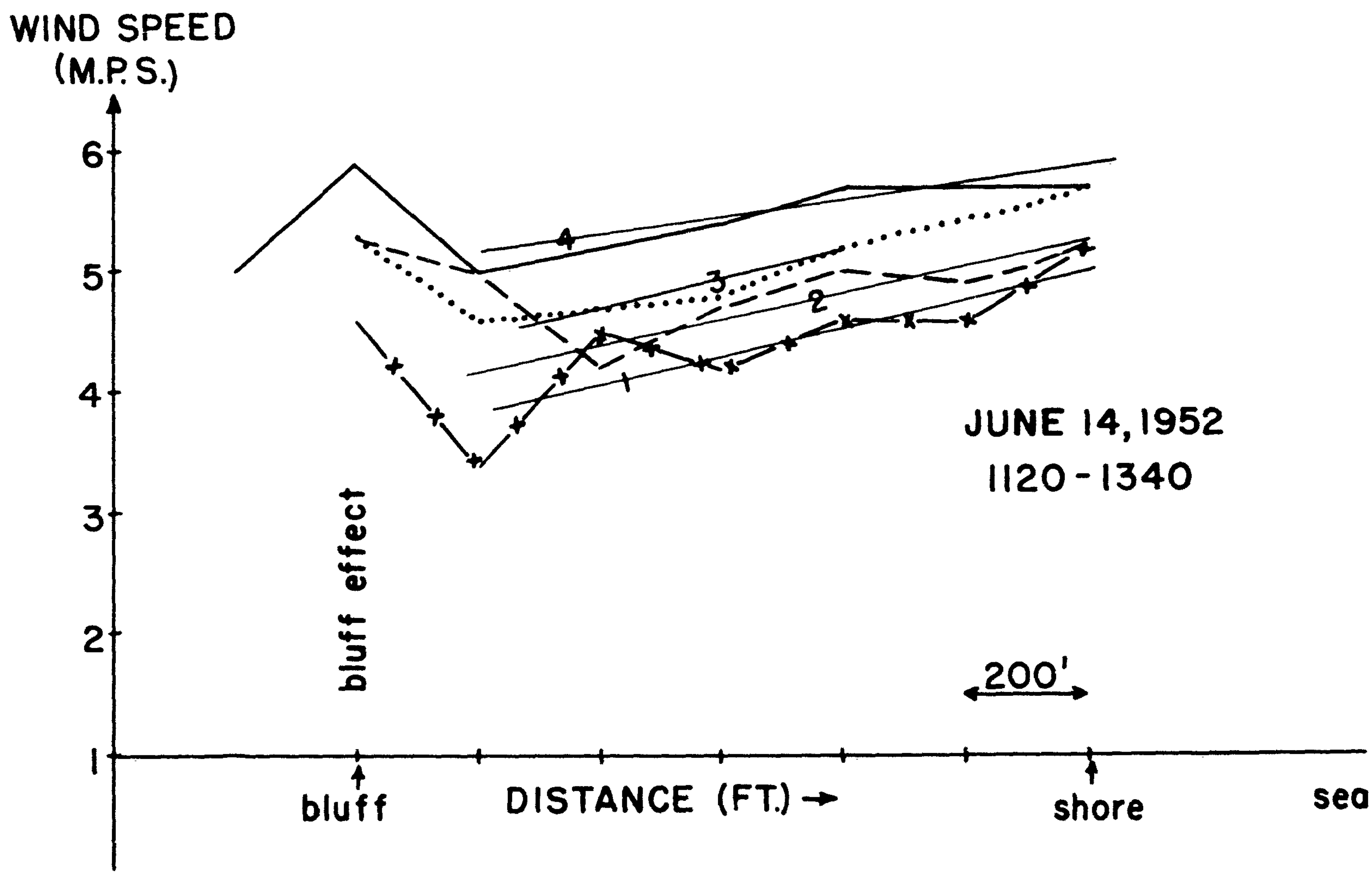


Figure 3 Observed horizontal profiles of the sea breeze at the windward shore of Nantucket Island.

with the theoretical relation (36), what is really wanted is the wind gradient from the shore out to sea rather than the one that was measured. Table 1 indicates that the measured quantity $\frac{\tau g}{\pi G T_m U}$ is constant within observational accuracy, fluctuating about a mean of two. The systematic deviation from the predicted value of unity is believed to be due to:

a) The ratio of the measured slope (on land where the sea breeze is decreasing with x) to the slope over the sea (where the sea breeze is increasing).

b) The non-uniformity of the island temperature.

A more intensive observational program is being considered by the meteorology group at this Institution which is based on the measurement of pressure differences to test (36). Thus, since $U\frac{\partial u'}{\partial \xi} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial \xi}$ equation (36) becomes

$$\frac{\bar{\rho}\tau g}{\pi W T_m} = 1$$

where

$\bar{\rho}$ = mean atmospheric density $\approx 1.2 \times 10^{-3}$ gms cm^{-3}

$g = 980$ cm sec^{-2}

τ = temperature excess of island at a given time

$-\frac{\partial p'}{\partial \xi} = W$ = perturbed pressure gradient at windward shore at a given time.

There seems to be little doubt that the pressure gradient associated with the sea breeze can be accurately measured (see for example, Leopold (1949)).

Table I

RESULTS OF SEA BREEZE MEASUREMENTS

Run No.	G cm/sec/800 ft.	τ °F	Angular Deviation of undisturbed wind from normal to coast	$\frac{\tau g}{\pi G T_m U}$
1	95	3.7	45°	1.8
2	90	5.0	30°	2.2
3	100	5.0	20°	1.7
4	60	5.7	0°	2.7

Computation of $\frac{\tau g}{\pi G T_m U}$ from observational measurements for comparison with theoretical results. U is the component of undisturbed gradient wind that is perpendicular to the coast, τ is the temperature excess of the island above the water, G is the horizontal gradient of the horizontal wind, $T_m = 300°C$, g = gravity, $\pi = 3.141$.

In concluding this section we present some conclusions on the turning and the hodograph of the sea breeze. This subject has received considerable theoretical and observational attention and it is generally accepted that the turning is due to the earth's rotation. The major purpose of the following development is to show how the various atmospheric parameters such as stability, eddy conductivity and undisturbed wind speed effect the hodograph. To compute v' equation (2) is used, and since the temperature gradient in the y direction vanishes, the y derivatives of the perturbation quantities are neglected. In dimensionless coordinates the y equation of motion becomes

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) v' = - c\, u'$$

The solution of this equation is

$$v'(x,t) = - c \int_{a}^{t} u'\left[x - t + \eta, \eta\right] \quad d\eta$$

where a is a constant of integration, as may be verified by differentiation. This will be evaluated at x = 0, which is the windward shore of the island. Now, u' varies as $\cos\lambda t$, hence at $t = -\frac{\pi}{2\lambda}$, $u' = 0$, and we assume $v' = 0$ at $t = t_o - \frac{\pi}{2\lambda}$, where t_o is small compared to $\frac{\pi}{2\lambda}$.

$$(37) \qquad \therefore \; \frac{v'(o,t)}{-c} = \int_{t_o - \frac{\pi}{2} - t}^{o} u'(\eta, \eta + t)\, d\eta$$

The value of u' is given by equation (35), but rather than attempt to integrate this complicated expression it is approximated by a simple exponential function which fits it in the vicinity of the windward shore. Thus, if u_o is the amplitude of u' given by (35) when $x = t = 0$ and G is the slope of this curve at the same point (i.e., $G = \frac{\partial u'}{\partial \xi}$) then,

$$u'(x,t) \approx u_o \exp \frac{x\, G}{u_o \sqrt{\frac{gs}{U^2}}} \cos \lambda t \qquad x < 0$$

where

$$u_o = \frac{g}{T_m} \frac{2}{\pi} \frac{K}{U^2} \quad \ln \frac{U^2}{K\sqrt{gs}}$$

$$G = \frac{\tau g}{\pi T_m U}$$

We seek the behavior of v' when $t \gg -\frac{\pi}{2\lambda} \sim -10^2$, i.e., well after the time of onset of the sea breeze. Since $\frac{x\, g}{u_o\sqrt{\frac{gs}{U^2}}} \ll 0$ when $x = t_o - \frac{\pi}{2} - t$ we may write (37) as

$$\frac{v'(o,t)}{-c\, u_o} = \text{Re} \int_{-\infty}^{o} e^{i\lambda t} \exp\left(\frac{\eta\, G}{u_o\sqrt{\frac{gs}{U^2}}} + i\lambda\eta\right) d\eta$$

$$(38) \quad \therefore \frac{v'(o,t)}{-c\, u_o} = \text{Re} \frac{e^{i\lambda t}}{\frac{G}{u_o\sqrt{\frac{gs}{U^2}}} + i\lambda}$$

where Re denotes "real part of".

From equation (38) it is seen that the sea breeze component parallel to the coast is, in general, not of the same phase as the diurnal temperature wave. The quantity $\frac{u_o\sqrt{\frac{gs}{U^2}}}{G}$ is a measure of the depth of the sea breeze, or how far out to sea it extends. When the depth is small compared with the distance that an air parcel travelling with the speed of the undisturbed wind would cover in $\frac{24}{2\pi}$ hours, then $v' = -c\, u_o \cos\lambda t$. In this case v' is much smaller ($\sim 10^{-2}$ times) than u' and the hodograph of the sea breeze components would be a straight line almost perpendicular to the coast. Figure 4 shows the theoretical hodograph when $\frac{G}{u_o\sqrt{\frac{gs}{U^2}}} = .05$, $\lambda = c = 10^{-2}$. This might correspond to a sea breeze that was five kilometers deep. It is readily shown from (38) that these hodographs are elliptic. The point labelled $\lambda t = 0$ in Figure 4 represents the time of maximum heating, i.e., the temperature difference between land and sea is a maximum. For deeper sea breezes, $\frac{G}{u_o\sqrt{\frac{gs}{U^2}}}$ decreases thereby increasing both the amplitude and phase angle of v', and hence the hodograph would tend to become more circular. These relations between the depth of the sea breeze and the shape of the hodograph at the coast would appear to have some interesting practical applications.

SUMMARY AND CONCLUSIONS

The first part of this paper deduces the equation of motion for a non-adiabatic atmosphere where the <u>mean</u> motions are small perturbations compared with the velocities in the basic current. The resulting partial differential equation is, aside from the non-homogeneous forcing function, the same as that for air flowing over a mountain.

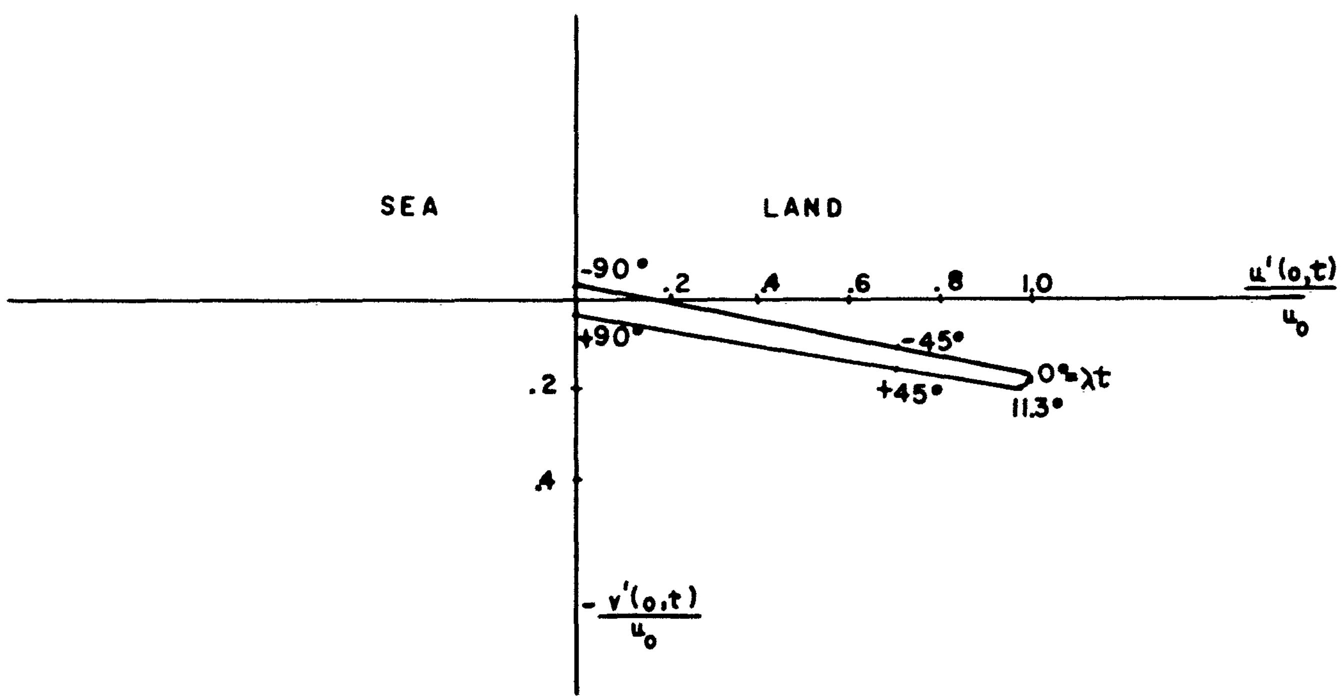


Figure 4 Theoretically computed hodograph of sea breeze components perpendicular to the coast (u') and normal to the coast (v'), when these are measured at windward shore. u_o is the maximum value of u'(o,t) and occurs when the island temperature reaches its maximum. The points placed on the hodograph represent the sea breeze vector at various times, converted to angular degrees. This hodograph might correspond to the sea breeze that is produced by a coast located in temperate latitudes under conditions of a light gradient wind, so that the seaward extent of the sea breeze was of the order of 5 km. (See p. 22, 23 for more details).

It is then sought to determine the mean perturbations that have their energy source in the turbulent heating near the ground. The inadequacy of the conventional formalism which describes the heating by means of an eddy conduction equation for the potential temperature is shown. Instead the heating is described in terms of H which is equal to the divergence of the eddy flux of heat, or to first order the total derivative of the potential temperature. On the basis of some physical arguements and hypotheses it is proposed that this function satisfies the equation

$$\frac{dH}{dt} = K \frac{\partial^2 H}{\partial z^2}$$

where $\frac{d}{dt}$ is the total hydrodynamic derivative and K is a mean value of the eddy conductivity for the <u>turbulent ground layer</u>.

The steady state solution of the equations of motion using a heating function that satisfies the above equations has been discussed by Stern and Malkus (1953) and compared with observations over Nantucket Island.

In order to further substantiate the theory this paper then proceeds to discuss the sea breeze by retaining the time derivatives and the Coriolis parameter where necessary on the scale pertaining to this phenomenon. It is shown that the sea breeze component perpendicular to the coast is in phase with the diurnal temperature wave and that it is not necessary to introduce friction to explain this. However, the sea breeze component parallel to the coast has a variable phase depending upon the depth of the sea breeze. The theory demonstrates the role of the various atmospheric parameters on the shape and size of the elliptic hodograph of the sea breeze components.

Preliminary observations of the sea breeze produced by a small flat island (Nantucket, Massachusetts) have indicated that quantitative as well as qualitative agreement of the theory may be expected. By means of this theory the sea breeze may be used as a fruitful tool to investigate the gross properties of turbulent heating near the ground, and it is hoped to be able to continue these observational studies.

The satisfactory picture of local convective phenomena that is produced by turbulent heating near the ground, would suggest that the theory now be applied to a larger scale. Preliminary theoretical investigations indicate that, despite obvious difficulties arising from assigning eddy conductivities on this scale, certain conclusions can be drawn which may be of heuristic value. This will constitute the subject of a future paper.

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